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Infinite- and finite-spin Ising chains at low temperatures

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Abstract. Transfer matrix methods are used to obtain the partition function and correlation length of Ising chains at low temperatures. For the infinite-spin chain, asymptotically exact eigenvalues and eigenvectors are obtained from an iteration scheme, and from a secular equation method. It is also shown how the Ising chain for general spin S (including $S \to \infty$) may be treated using a 2×2 transfer matrix.

For low temperatures ($K \equiv J/K_B T \gg 1$) the inverse correlation length, in units of the spin spacing, is

$$\xi^{-1} = 4S\, e^{-2K} + O(e^{-4K}) \qquad \text{(finite } S\text{)}$$

$$= 8K\, e^{-2K}\left(\ln 4K + \gamma + O\left(\frac{\ln K}{K}\right)\right) \qquad \text{(infinite spin)}$$

where γ is Euler's constant.

1. Introduction

The infinite- ('classical') and finite-spin Ising chains considered in this paper are among the simplest interacting spin models. As is well known (Stanley 1971, Domb 1960) they can be treated using commuting transfer matrices whose largest eigenvalues give the partition function and correlation length.

These eigenvalues can be easily found for the $S \times S$ matrices occurring for low finite spin S (Suzuki *et al* 1967). For general or infinite S the problem is much more difficult. The classical model has been treated for zero field by Thompson (1968, 1972) and, as the limit of the anisotropic Heisenberg model, by Joyce (1967a, b) and by Rae (1974, 1975a, b). From this work it is known that at low temperatures the correlation length ξ (in units of the spin spacing) has the form

$$\xi^{-1} = f(K)\, e^{-2K} \qquad K \gg 1 \tag{1}$$

where (in terms of the exchange constant J and temperature T)

$$K = J/K_B T \tag{2}$$

and $f(K)$ is a prefactor estimated by Thompson (1968) as about $25K$ for K between 5 and 8. In this paper it will be shown that

$$\xi^{-1} = 8K\left(\ln 4K + \gamma + O\left(\frac{\ln K}{K}\right)\right) e^{-2K} \tag{3}$$

where $\gamma = 0{\cdot}577$ is Euler's constant. The prefactor in (3) agrees with Thompson's numerical estimate for K in (5, 8).

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The result (3) may be arrived at by exploiting the properties of the integral equation for the eigenvalues of the transfer matrix in any of three ways. The first is an iteration scheme, which converges rapidly because of the dominance of the two largest eigenvalues which are associated with eigenfunctions of different symmetry. The second method generates asymptotic expansions for the eigenfunctions, which are then inserted into an exact result for the difference of the two largest eigenvalues. This method also gives as the solution of a truncated $n \times n$ determinantal secular equation the first n terms in an asymptotic expansion for the largest eigenvalue λ_1. The first five terms are

$$\lambda_1 = \frac{e^K}{2K}\left(1 + \frac{1}{4K} + \frac{3}{16K^2} + \frac{7}{32K^3} + \frac{85}{256K^4} + O\left(\frac{1}{K^5}\right)\right). \tag{4}$$

The Nth power of λ_1 is the partition function Z of the N-spin chain.

The third method of obtaining (3) uses a trick which reduces the problem to the consideration of a 2×2 transfer matrix. This reduction holds also for arbitrary finite spin S, where it leads to the following result for the inverse correlation length

$$\xi^{-1} = 4S\, e^{-2K} + O(e^{-4K}), \tag{5}$$

which is in agreement with a result of Sneddon and Stinchcombe (1978).

The paper is laid out as follows. Section 2 gives the basic properties of the system and of the eigenvalue problem, leading to the iteration scheme and to the exact result for the difference of the two largest eigenvalues. Section 3 gives the eigenvalues for the infinite-spin chain from the iteration scheme and from the exact result and asymptotic expansion of the eigenfunctions. Section 4 uses the 2×2 matrix method to obtain the correlation length of the finite-spin chain and makes contact with the results of the preceding sections for the correlation length of the infinite-spin chain.

2. Basic properties

The Ising chain has Hamiltonian

$$H = -J \sum_{n=1}^{N} \sigma_n \sigma_{n+1} \qquad \sigma_{N+1} = \sigma_1 \tag{6}$$

where σ_n is the z component of the (normalised) spin at site n. The transfer matrix T has as its elements the probabilities of configurations $(\sigma_n, \sigma_{n+1}) = (x, y)$ of neighbouring spins

$$T_{xy} = e^{Kxy} \tag{7}$$

with K given by (2). If $\lambda_1, \lambda_2, \ldots$ label the eigenvalues of T in order of decreasing size, and $\psi_1, \psi_2, \ldots$ are the corresponding eigenfunctions, it can be shown that the partition function and longest correlation length of long chains ($N \to \infty$) are given by

$$Z = \lambda_1^N \tag{8}$$

$$\xi = (-\ln(\lambda_2/\lambda_1))^{-1}. \tag{9}$$

The symmetry properties of (7) (invariance under $(x, y) \to (y, x)$ or $(-x, -y)$, etc) imply that the eigenfunctions are of even or odd parity. The eigenvalue equation for the even and odd functions (ψ^+ and ψ^-) can then be reduced to

$$\lambda^{\pm}\psi^{\pm}(x)=\int_0^1 \mathrm{d}y(\mathrm{e}^{Kxy}\pm \mathrm{e}^{-Kxy})\psi^{\pm}(y)\equiv \mathscr{L}^{\pm}\psi^{\pm}. \tag{10}$$

Except for a few obvious exceptions towards the end of this paper we work hereafter with spin projections x, y only in the positive interval (0, 1).

In (10), and for the rest of this section, we write integral forms appropriate to the infinite-spin case. However, all the formal conclusions of this section apply equally to the finite-spin case provided only that any integral, such as that in (10), is replaced by a sum over discrete non-negative spin projections ranging up to 1 through steps of size $1/S$.

Each of the linear operators $\mathscr{L}^{\pm}$ in (10) satisfies the conditions for their having complete sets of orthogonal eigenfunctions in the interval (0, 1). Thus if ψ_a^+, ψ_b^+ are two eigenfunctions of $\mathscr{L}^+$ associated with non-degenerate eigenvalues λ_a^+, λ_b^+,

$$(\psi_a^+,\psi_b^+)\equiv\int_0^1 \mathrm{d}x\,\psi_a^+(x)\psi_b^+(x)=0 \tag{11}$$

(similarly for eigenfunctions of $\mathscr{L}^-$). It also follows from (10) that

$$\lambda_a^- - \lambda_b^+ = \frac{(\psi_b^+,(\mathscr{L}^- - \mathscr{L}^+)\psi_a^-)}{(\psi_b^+,\psi_a^-)}. \tag{12}$$

These exact statements will be useful in later developments.

The essential physical point to be exploited is that the Ising chain is critical (i.e. its correlation length diverges) at $T=0$. Thus, from (9), in the low-temperature regime in which we are interested ($K\gg 1$, which is assumed throughout this paper) the two largest eigenvalues λ_1, λ_2 have to be very nearly degenerate. Since no crossover phenomena are expected in the critical behaviour we can also suppose (following Riedel and Wegner (1969) and Riedel (1971)) that all other eigenvalues will be much smaller. The nature of the kernels in (10) ($\exp(Kxy)\gg\exp(-Kxy)>0$ for x, y both positive) suggests that the nearly degenerate eigenvalues λ_1, λ_2 are associated respectively with even and odd eigenfunctions (hereafter written in either of the equivalent forms ψ_1 or ψ_1^+, ψ_2 or ψ_2^-). All of these expectations are borne out by subsequent calculations.

So, if the integral operator $\mathscr{L}^+$ is applied n times to an arbitrary linear combination ϕ_0^+ of even eigenfunctions (i.e. to an even function ϕ_0^+) we obtain

$$(\mathscr{L}^+)^n\phi_0^+\equiv(\mathscr{L}^+)^n\sum_r c_r\psi_r^+=\sum_r c_r(\lambda_r^+)^n\psi_r^+\equiv\phi_n^+. \tag{13}$$

Because λ_1 is much larger than any other λ_r^+, ϕ_n^+ rapidly approaches the eigenfunction ψ_1^+ associated with the largest eigenvalue λ_1, except in freak cases where ϕ_0^+ has no overlap ($c_0=0$) with the required eigenfunction. This is the basis of the iteration scheme.

It is also easy to deduce the inequality

$$(\phi_n^+,\mathscr{L}^+\phi_n^+)/(\phi_n^+,\phi_n^+)\leq\lambda_1. \tag{14}$$

Thus the ratio on the left-hand side of (14) provides a lower bound for the largest eigenvalue λ_1 and can be used to set up a variational or iterative attack on the problem (see also McGurn and Scalapino 1975). A more convenient form for the iterative approach is to use the sequence

$$\phi_{n+1}^{+}(1)/\phi_{n}^{+}(1) \equiv \lambda_{1}^{(n)}, \tag{15}$$

which converges quickly to the largest eigenvalue λ_1.

Similarly, application of the integral operator $\mathscr{L}^{-}$ to an odd seed function ϕ_0^{-} generates a rapidly convergent sequence $\{\phi_n^{-}\}$ of approximations to the odd eigenfunction ψ_2^{-} and through

$$\phi_{n+1}^{-}(1)/\phi_{n}^{-}(1) \equiv \lambda_{2}^{(n)} \tag{16}$$

to the second eigenvalue λ_2.

In the infinite-spin case we can formally identify, from Joyce (1967a), λ_1 and λ_2 with the radial spheroidal wavefunctions $\mathrm{i}^l je_{|m|l}(-\mathrm{i}K, 1)$ for $(m, l) = (0, 0)$ and $(0, 1)$ respectively (the notation is that of Morse and Feshbach 1953). The iteration scheme thus provides a tractable and quickly convergent sequence of approximations to these quantities for large K.

3. Eigenfunctions and eigenvalues for low-temperature infinite-spin chains

The iteration scheme just described, starting from the seed functions

$$\phi_0^{\pm} = (\delta(x-1) \pm \delta(x+1)), \tag{17}$$

yields the following sequence of approximations for the ratio of the largest eigenvalues of the infinite-spin chain:

$$(\lambda_2/\lambda_1)^{(1)} = 1 - 8K\,\mathrm{e}^{-2K} + \ldots \tag{18}$$

$$(\lambda_2/\lambda_1)^{(2)} = 1 - 8K\,\mathrm{e}^{-2K}(\ln K + \mathrm{O}(1)) \tag{19}$$

$$(\lambda_2/\lambda_1)^{(3)} = 1 - 8K\,\mathrm{e}^{-2K}\left(\ln 4K + \gamma + \mathrm{O}\left(\frac{\ln K}{K}\right)\right) \tag{20}$$

where $\gamma = 0{\cdot}577$ is Euler's constant. Equation (20) leads at once to the result (3) for the inverse correlation length. The asymptotic evaluation of integrals contributing to (19), (20) and to later results is outlined in appendix 1. We discuss the reasons for the quoted errors elsewhere.

The sequence of approximations to the eigenfunctions ψ_1^{+}, ψ_2^{-} generated from (17), or from any other reasonable even and odd initial functions, suggests the following forms for the eigenfunctions:

$$\psi^{\pm}(x) = \mathrm{e}^{Kx}\left(\sum_{n=0} c_n^{\pm}(K)\left(\frac{1-x}{2}\right)^n + \mathrm{O}(\mathrm{e}^{-2K})\right) \tag{21}$$

(for convenience we omit the subscripts 1, 2 on ψ^{+}, ψ^{-}). By inserting these forms into (10) we can find self-consistent equations for the $c_n^{\pm}$ and for the eigenvalues $\lambda^{+}(=\lambda_1)$, $\lambda^{-}(=\lambda_2)$. If we neglect terms smaller by $\mathrm{O}(\mathrm{e}^{-2K})$ than the leading one, λ^{-}

and c_n^- satisfy the same equations as λ^+ and c_n^+, so

$$c_n^+ = c_n^-(1+\mathrm{O}(\mathrm{e}^{-2K})) \tag{22}$$

$$\lambda^+ = \lambda^-(1+\mathrm{O}(\mathrm{e}^{-2K})). \tag{23}$$

The difference of λ^- from λ^+ comes, of course, from such neglected terms but, as will be discussed later, we can nevertheless arrive at the difference by using the eigenfunctions (21), without the exponentially small corrections, in the exact result (12).

The self-consistent equations resulting from (21) and (10) yield a secular equation for the eigenvalues $\lambda^\pm$, which gives a very convenient way of arriving at results like (4). We now outline this procedure.

Inserting (21) into (10), neglecting terms smaller than the leading term by $\mathrm{O}(\mathrm{e}^{-2K})$ and equating coefficients of $[\frac{1}{2}(1-x)]^m$ gives

$$\mu c_m = \sum_{n=0} c_n \epsilon^n (n+m)!/m! \equiv \sum_{n=0} M_{mn} c_n \tag{24}$$

where

$$\mu \equiv \lambda\, \mathrm{e}^{-K} 2K \qquad \epsilon \equiv 1/4K. \tag{25}$$

The quantities μ are the eigenvalues of the matrix M defined by (24). The secular equation determining them has the form

$$0 = \begin{vmatrix} 1-\mu & \epsilon & 2\epsilon^2 & 6\epsilon^3 & \cdots \\ 1 & 2\epsilon-\mu & 6\epsilon^2 & 24\epsilon^3 & \\ 1 & 3\epsilon & 12\epsilon^2-\mu & 60\epsilon^3 & \\ 1 & 4\epsilon & 20\epsilon^2 & 120\epsilon^3-\mu & \\ \cdots & & & & \end{vmatrix}. \tag{26}$$

Since columns of M fall off in size like ϵ^n, to arrive at μ to order $(1/K)^{n-1}$ it is only necessary to consider the top left-hand $n \times n$ corner block of the secular determinant, as can be seen by considering its expansion by a column outside of the $n \times n$ corner. For example, the 5×5 block yields for the largest eigenvalue

$$\mu = 1 + \epsilon + 3\epsilon^2 + 14\epsilon^3 + 85\epsilon^4 + \mathrm{O}(\epsilon^5) \tag{27}$$

which implies the result (4) for the Nth root of the partition function.

It is easy to show that the corresponding eigenvector has

$$c_m = c_0(1+\mathrm{O}(1/K)) \qquad \text{for any } m. \tag{28}$$

The secular equation (26) has other solutions for μ besides the largest one given in (27). These correspond to the odd and even eigenvalues (not split in this approximation) other than λ_1^+, λ_2^-, and they are of order ϵ which confirms the assumption that at low temperatures all other eigenvalues are much smaller than λ_1^+, λ_2^-.

Details of the method of arriving at the splitting of the eigenvalues, by putting the asymptotic expansions (21) and (28) for the eigenfunctions into the exact result (12), are given in appendix 2. This gives a second way of arriving at (20) and the result (3) for the correlation length, and shows clearly the origin of the errors quoted in (20).

In the next section we consider the finite-spin chain, using a method which allows contact to be made with the above results for the infinite-spin chain and with the standard method for the spin-$\frac{1}{2}$ case.

4. Relationship between spin-$\frac{1}{2}$, spin S, and infinite-spin correlation lengths

We now consider general spin S, with the σ's still normalised to unity. For large separations r, the correlation function is

$$\langle\sigma_1\sigma_{1+r}\rangle = \Sigma_{12}^2(\lambda_2/\lambda_1)^r. \tag{29}$$

λ_2 and λ_1 are again the next and largest eigenvalues of the transfer matrix T which is now a discrete $(2S+1)\times(2S+1)$ matrix with elements

$$T_{kl} = \exp(Kkl) \tag{30}$$

where k, l run over the $2S+1$ values $(-1+r/S)$ where r is any integer from zero to $2S$. In terms of the matrix U of normalised column eigenvectors of T, the matrix Σ is

$$\Sigma = U^{\mathrm{T}}\sigma U. \tag{31}$$

Equation (29) implies the relationship (9).

The general spin case can be discussed in terms of a 2×2 transfer matrix by replacing every rth spin σ in the spin-S chain by a spin-$\frac{1}{2}$ (μ) as shown in figure 1. The resulting

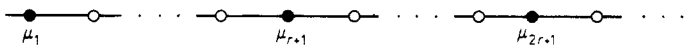

Figure 1. Construction for the discussion of the spin-S chain in terms of a 2×2 transfer matrix. Every rth spin has been replaced by a spin $\frac{1}{2}$.

2×2 transfer matrix $\mathcal{T}$ between adjacent spins μ has elements

$$\mathcal{T}_{\mu_1\mu_{r+1}} = \sum_{kl} \exp(K\mu_1 k)(T^{r-2})_{kl} \exp(Kl\mu_{r+1}) \tag{32}$$

where $\mu_1(=\pm1)$ and $\mu_{r+1}(=\pm1)$ label the states of the two spins $\frac{1}{2}$. It can be seen that

$$\mathcal{T}_{11} = \mathcal{T}_{-1,-1} = \sum_{kl} T_{1k}(T^{r-2})_{kl}T_{l1} = (T^r)_{11} \tag{33}$$

$$\mathcal{T}_{1,-1} = \mathcal{T}_{-1,1} = \ \ldots\ = (T^r)_{1,-1}. \tag{34}$$

We now show that for large r

$$\langle\mu_1\mu_{r+1}\rangle = A(K)\langle\sigma_1\sigma_{r+1}\rangle \tag{35}$$

where $A(K)$ is a constant independent of r. This is because, with t_1, t_2 the two eigenvalues of $\mathcal{T}$,

$$\langle\mu_1\mu_{r+1}\rangle = \frac{t_2}{t_1} = \frac{\mathcal{T}_{11}-\mathcal{T}_{1,-1}}{\mathcal{T}_{11}+\mathcal{T}_{1,-1}} = \frac{(T^r)_{11}-(T^r)_{1,-1}}{(T^r)_{11}+(T^r)_{1,-1}}. \tag{36}$$

But

$$T^r = U\Lambda^r U^{\mathrm{T}} \tag{37}$$

where Λ is the matrix of eigenvalues of T, and for large r (37) is dominated by the contribution from the two largest eigenvalues λ_1, λ_2. The associated eigenvectors occurring in U are respectively of even and odd parity so, for large r,

$$\langle(\mu_1\mu_{r+1}\rangle = (U_{12}/U_{11})^2(\lambda_2/\lambda_1)^r \tag{38}$$

which, because of (29), implies the result (35) and in particular, using (36)

$$\lambda_2/\lambda_1 = \lim_{r\to\infty} (t_2/t_1)^{1/r}. \tag{39}$$

The ratio of eigenvalues can now be worked out readily from (36). As $K \to \infty$, the leading term in $\mathcal{T}_{11}$ is $(T_{11})^r = e^{rK}$. The leading term in $\mathcal{T}_{1,-1}$ takes the general form

$$T_{11}T_{11} \ldots T_{ij}T_{j,-1}T_{-1,-1} \ldots T_{-1,-1}.$$

There are $2S(r-1)$ such terms, each of which is $e^{rK} e^{-2K}$. Thus

$$t_2/t_1 = 1 - 4S(r-1)\, e^{-2K} + O(e^{-4K}), \tag{40}$$

so that, using (39),

$$\lambda_2/\lambda_1 = 1 - 4S\, e^{-2K} + O(e^{-4K}). \tag{41}$$

The inverse correlation length of the finite-spin system is therefore as given in (5). This agrees with a result of Sneddon and Stinchcombe (1978), the proof of which also indicated a relationship, which (35) makes precise, between spin-$\frac{1}{2}$ and general-spin systems.

The ideas of the above method, and in particular the results (35), (36) and (39), also apply in the case of infinite spin. Here, of course, sums become integrals and in place of (32) we have

$$\mathcal{T}_{\mu_1\mu_{r+1}} = \int_{-1}^{1} dx_2 \ldots \int_{-1}^{1} dx_r \exp[K(\mu_1 x_2 + x_2 x_3 + \ldots + x_r \mu_{r+1})]. \tag{42}$$

We can proceed, as above, to work out λ_2/λ_1 using (39) and (36), which requires the evaluation, for large K and large r, of $\mathcal{T}_{11}$ and $\mathcal{T}_{1,-1}$. This calculation is most easily carried out (appendix 3) by using (42) for the case r even and integrating over the variables x_i with i even and then asymptotically evaluating the remaining integrals by methods like those already used in appendices 1 and 2. The results are (ignoring exponentially small corrections)

$$\mathcal{T}_{11} = (e^K/2K)^{r-2} \tag{43}$$

$$\mathcal{T}_{1,-1} = \left(\frac{e^K}{2K}\right)^{r-2} 4K\, e^{-2K}\left(\ln 4K + \gamma + O\left(\frac{\ln K}{K}\right)\right)(r + O(1)). \tag{44}$$

Using (36) and (39) we again recover the result (20) and hence the result (3) for the correlation length of the infinite-spin system:

$$\xi^{-1} = 8K\left(\ln 4K + \gamma + O\left(\frac{\ln K}{K}\right)\right) e^{-2K}. \tag{45}$$

5. Summary

The principal results we have obtained, namely the correlation length and partition function of the finite-spin Ising chain and the correlation length of the finite-spin chain, have already been quoted as equations (3), (4) and (5) of the introduction. They are valid in the low-temperature limit.

The most difficult result to obtain, that for the infinite-spin correlation lengths, has been arrived at by three different methods. The first, the iterative method, is set up so that it must converge quickly. The method exploiting the exact result (12) is not sensitive to the difference between the nearly degenerate eigenfunctions. We constructed a method for arriving at the asymptotic expansion of these functions, but only exploited the resulting secular equation to obtain the long series (4) for the partition function. The third method for arriving at the correlation length proceeds by relating the correlation functions of spin-$\frac{1}{2}$ and arbitrary spin systems. As well as recovering the correlation lengths of the infinite-spin chain we obtain it for general finite spin S, making contact with results of Sneddon and Stinchcombe (1978).

The methods have individual advantages which may make one of them more suited than the others in particular sensitive systems. The classical anisotropic Heisenberg model is presently being considered from this point of view.

Appendix 1. Asymptotic evaluation of integrals occurring in the iteration approach

When $\mathscr{L}^+$ and $\mathscr{L}^-$ are applied to the seed functions (17) we obtain

$$\phi_1^{\pm}(x) = e^{Kx} \pm e^{-Kx} \tag{A.1}$$

$$\phi_2^{\pm}(x) = \frac{2\sinh K(1+x)}{K(1+x)} \pm \frac{2\sinh K(1-x)}{K(1-x)}. \tag{A.2}$$

Insertion into (15) and (16) gives the result (18) for $\lambda_2^{(1)}/\lambda_1^{(1)}$. The higher iterations required to obtain (19) and (20) are

$$\phi_3^{\pm}(1) = \int_{-1}^{1} dy(e^{Ky} \pm e^{-Ky})\frac{2\sinh K(1+y)}{K(1+y)} \tag{A.3}$$

$$\phi_4^{\pm}(1) = \tfrac{1}{2}\int_{-1}^{1} dy(\phi_2^{\pm}(y))^2. \tag{A.4}$$

For large K, $\phi_3^{\pm}(1)$ are dominated by the following parts

$$\phi_3^{\pm}(1) \approx \frac{e^K}{K}(I_1 \pm I_2 \pm \ln 2) \tag{A.5}$$

where

$$I_1 \equiv \int_0^1 dy \frac{e^{2Ky}}{1+y} = \frac{e^{2K}}{4K}\left(1 + O\left(\frac{1}{K}\right)\right) \tag{A.6}$$

$$I_2 \equiv \int_0^1 ds(1 - e^{-2Ks})/s = \ln 2K + \gamma + O(e^{-K}), \tag{A.7}$$

the evaluation of such integrals being carried out by the usual Laplace method of asymptotic analysis as given, for example, by de Bruijn (1958) and Murray (1974). γ is Euler's constant. These results are already sufficient to give (19).

The dominant contributions to $\phi_4^{\pm}(1)$ are

$$\phi_4^{\pm}(1) \approx \frac{e^{2K}}{K^2}(I_3 \pm I_4) \tag{A.8}$$

where

$$I_3 \equiv \int_0^1 dy \frac{e^{2Ky}}{(1+y)^2} = \frac{e^{2K}}{8K}\left(1+O\left(\frac{1}{K}\right)\right) \tag{A.9}$$

$$I_4 \equiv \int_0^1 ds \frac{(1-e^{-2Ks})}{s(1-\frac{1}{2}s)} = \left(\ln 4K + \gamma + O\left(\frac{1}{K}\right)\right). \tag{A.10}$$

Equation (20) follows in a straightforward manner from these results.

Appendix 2. Derivation of the correlation length using an exact result for the eigenvalue splitting

Insertion of the asymptotic expansion (20) for the eigenfunctions into the exact result (12) gives the following expression for the eigenvalue splitting:

$$\lambda_2 - \lambda_1 = \frac{-2\Sigma_{nm} c_n c_m 2^{-(n+m)} e^{-K} \int_0^1 ds \int_0^1 dt\, s^n t^m e^{-Kst} + \dots}{\Sigma_{nm} c_n c_m 2^{-(n+m)} \int_0^1 ds\, e^{-2Ks} s^{n+m} + \dots} \tag{A.11}$$

$$= -2\, e^{-K}(N_1+N_2)/D \tag{A.12}$$

where

$$N_1 \equiv \sum_m c_m^2 2^{-2m} (\partial/\partial(-K))^m J_1(K) \tag{A.13}$$

$$N_2 \equiv \sum_{n>m} c_n c_m 2^{-(n+m)} (\partial/\partial(-K))^m J_2(K) \tag{A.14}$$

$$D \equiv \sum_{nm} c_n c_m 2^{-(n+m)} (\partial/\partial(-K))^{n+m} \left(\frac{1-e^{-2K}}{2K}\right). \tag{A.15}$$

These expressions involve the integrals

$$J_1(K) \equiv \int_0^1 ds \frac{(1-e^{-Ks})}{Ks} = \frac{1}{K}[\ln K + \gamma + O(e^{-K})] \tag{A.16}$$

$$J_2(K) \equiv \int_0^1 ds \frac{s^{n-m-1}}{K}(1-e^{-Ks})$$

$$= \left(\frac{1}{(n-m)K} - \frac{(n-m-1)!}{K^{n-m+1}} + O(e^{-K})\right) \qquad (n>m). \tag{A.17}$$

Using (28) and discarding exponentially small terms we then obtain

$$\lambda_2 - \lambda_1 = \frac{-2\, e^{-K}[K^{-1}(\ln K + \gamma) + 2\Sigma_{n>0}(nK)^{-1} + O[(\ln K)/K^2]]}{\frac{1}{2}K^{-1} + O(K^{-2})} \tag{A.18}$$

$$= -4e^{-K}\left(\ln 4K + \gamma + O\left(\frac{\ln K}{K}\right)\right). \tag{A.19}$$

Dividing by λ_1 (which is given by (25) and (27)) we thus recover the result (20).

The error quoted in (A.19) can be verified by a careful consideration of all the terms discarded to obtain the result. The discarded terms include exponentially small terms already omitted from (A.11) (which include those terms omitted in (21) which would be required to make $\psi^{\pm}$ have the appropriate symmetry). The leading corrections come from the $m = 1$ term in N_1.

Appendix 3. Transfer matrix between spins $\frac{1}{2}$ in a classical Ising chain

We here evaluate the two independent elements $\mathcal{T}_{11}$, $\mathcal{T}_{1,-1}$ of the transfer matrix (42) for two spins-$\frac{1}{2}$ (μ_1, μ_{r+1}) separated by $r-1$ classical spin vectors with z components $x_2, \ldots, x_r$.

We suppose that r is even (equal to $2m$) and integrate over $x_2, x_4, \ldots x_{2m}$ to obtain

$$\mathcal{T}_{\mu_1\mu_{r+1}} = \int_{-1}^{1} dx_3 \ldots \int_{-1}^{1} dx_{r-1} \mathcal{S}(\mu_1, x_3)\mathcal{S}(x_3, x_5) \ldots \mathcal{S}(x_{r-1}, \mu_{r+1}) \tag{A.20}$$

where

$$\mathcal{S}(x, y) = \frac{2 \sinh K(x+y)}{K(x+y)}. \tag{A.21}$$

The greatest contribution to $\mathcal{T}_{11}$ will occur when all the x's are near one. We therefore discard the negative half of each integral and define $z = 1 - x$. The following types of term arise:

$$\frac{2 \sinh K(2 - z_3 - z_5)}{2 - z_3 - z_5} \sim \tfrac{1}{2} \exp(2K - z_3 K - z_5 K)[1 + \tfrac{1}{2}(z_3 + z_5) + \ldots]. \tag{A.22}$$

Retaining only the leading term in the expansion, these can be easily integrated to yield

$$\mathcal{T}_{11} = (e^{2K}/4K^2)^{m-1}(1 + O(K^{-1})). \tag{A.23}$$

Next we consider the leading terms in $\mathcal{T}_{1,-1}$. There will be $m-3$ terms where the first l x's are in the range (0, 1) and the last $m-l-1$ are in the range $(-1, 0)$ (with l and $m-l-1$ both non-zero), plus the two 'end' terms $l = 0$ and $l = m-1$. The contribution from the 'end' terms need not be considered further since we shall eventually take r large (therefore m large) to obtain λ_2/λ_1 using equation (39). In each of the $m-3$ other terms put $x = 1 - z$ for the first l x's and $x = -(1-z)$ for the rest. To compare $\mathcal{T}_{1,-1}$ with $\mathcal{T}_{11}$ we then have to compare, for each of the $m-3$ terms,

$$J_3 \equiv \int_0^1 dz_l \int_0^1 dz_{l+1} \frac{e^{-Kz_l} 2 \sinh K(z_{l+1} - z_l)\, e^{-Kz_{l+1}}}{[1 - \frac{1}{2}(z_{l-1} + z_l)]K(z_{l+1} - z_l)[1 - \frac{1}{2}(z_{l+1} + z_{l+2})]} \tag{A.24}$$

with

$$J_4 \equiv \int_0^1 dz_l \int_0^1 dz_{l+1} \frac{e^{-Kz_l} 2 \sinh K(2 - z_l - z_{l+1})\, e^{-Kz_{l+1}}}{[1 - \frac{1}{2}(z_{l-1} + z_l)]K(2 - z_l - z_{l+1})[1 - \frac{1}{2}(z_{l+1} + z_{l+2})]} \tag{A.25}$$

$$= \frac{e^{2K}}{(2K)^3}\left(1 + O\left(\frac{1}{K}\right)\right) \tag{A.26}$$

(where the subscripts are now such that $z_l = 1 - x_{2l+1}$, etc). Note that in (A.24) and (A.25) we can discard each z_{l-1} and z_{l+2} in the the denominators since they are each associated with a factor $1/K$ by virtue of the exponents $e^{-2Kz_{l-1}}$, $e^{-2Kz_{l+2}}$ in the integrals over z_{l-1}, z_{l+2}. Changing variables to $u = z_{l+1} - z_l$, $v = z_{l+1} + z_l$ and expanding the denominators $(1 - \frac{1}{2}z_l)$, $(1 - \frac{1}{2}z_{l+1})$ (A.24) becomes

$$J_3 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\alpha=0}^{n} \sum_{\beta=0}^{m} 4^{-(n+m)}(-1)^{\alpha}\, {}_nC_{\alpha}\, {}_mC_{\beta}\, I_{\alpha+\beta, n+m-\alpha-\beta} \tag{A.27}$$

where

$$I_{n_1,n_2} \equiv \int_{-1}^{1} \mathrm{d}u \int_{|u|}^{2-|u|} \mathrm{d}v\ \mathrm{e}^{-vK} \frac{\sinh Ku}{Ku} u^{n_1} v^{n_2} \tag{A.28}$$

$$= I_{n_1+n_2,0} + \mathrm{O}(1/K^3) \qquad (n_1, n_2) \neq (0, 1) \tag{A.29}$$

$$I_{0,1} = I_{1,0} + \mathrm{O}(\ln K/K^3). \tag{A.30}$$

Using (A.29) and (A.30) the only integral needed to evaluate (A.27) is

$$I_{n',0} = \frac{1}{K^2}\left(\ln 2K + \gamma + \mathrm{O}\left(\frac{1}{K}\right)\right) \qquad n' = 0 \tag{A.31}$$

$$= \frac{1}{K^2}\left(\frac{1}{n'} + \mathrm{O}(\mathrm{e}^{-K})\right) \qquad n' \geq 1. \tag{A.32}$$

Thus, from the $(m-3)$ terms, we find the result

$$\frac{\mathcal{T}_{1,-1}}{\mathcal{T}_{1,1}} = (m + \mathrm{O}(1))\frac{J_3}{J_4} = 4K\ \mathrm{e}^{-2K}\left(\ln 4K + \gamma + \mathrm{O}\left(\frac{\ln K}{K}\right)\right)(r + \mathrm{O}(1)). \tag{A.33}$$

Equations (A.23) and (A.33) yield the results (43) and (44) quoted in the text, which lead to the correlation length (45).

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